JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 23, No. 2, March–April 2000

Design of Cascade Compensators Using Generalized Singular Values

Mario A. Rotea
Purdue University, West Lafayette, Indiana 47907-1282

When active control is used to improve plant dynamics, cascade compensators (or loop shaping filters) are often used to help direct the available control energy to the particular subset of plant modes whose dynamics is to be modified. In certain applications the cascade compensators can be taken to be simple frequency selective filters such as low-pass or bandpass filters. There are applications in which this is not the case. A method is described to obtain cascade compensators that achieve the stated goal when frequency selective filters do not suffice to obtain a solution. The computations required to implement the results are rather simple and involve the calculation of the generalized singular value decomposition of a matrix pair constructed with frequency response data. When combined with robust methods for controller design, the results yield a simple and effective method for designing controllers that modify the dynamics associated with a specified subset of plant poles. A detailed numerical example is included to illustrate ideas and computations.

Nomenclature

$A \geq B (A > B)$	= $A - B$ has (strictly) positive eigenvalues only, $A = A^*$ and $B = B^*$
A'	= transpose of matrix A
A^*	= conjugate transpose of matrix A
$\mathrm{diag}(d_1, \ldots, d_n)$	= rectangular matrix with entries a_{ij} satisfying $a_{ii} = d_i$, and $a_{ij} = 0$ whenever $i \neq j$
I	= identity matrix
$\Im\{\cdot\}$	= imaginary part
im A	= range space of matrix A
$\Re\{\cdot\}$	= real part
$\sigma_j(M_1, M_2)$	= jth generalized singular value of the matrix pair M_1, M_2; see Appendix for its definition
$\sigma_{\max}(M_1, M_2)$	= largest generalized singular value
$\|\cdot\|$	= Euclidean norm

I. Introduction

WHEN active control is used to improve plant dynamics, cascade compensators (or loop shaping filters) are often used to help direct the available control energy to the particular subset of plant modes whose dynamics is to be modified. These cascade compensators should also minimize the energy spillover to the less important modes. For example, in the control of lightly damped systems, it is common practice to add, in cascade with the plant, frequency selective filters with passing bands that include the plant modes whose damping is to be increased and exclude any other plant modes. Such filters not only help focus the control energy on the relevant modes but also help reduce the interaction of the control signals with any modes outside the band of interest.

When there is frequency separation, between the poles we wish to control and the other plant poles, the design of such cascade compensators is fairly simple. Indeed, due to the frequency separation, the cascade compensators can be taken to be conventional low-pass, high-pass, or bandpass filters, depending on the frequency band(s) occupied by the plant poles whose dynamics is to be improved. These frequency selective filters make the in-band plant poles more controllable than the out-of-band plant poles. This property is achieved by increasing the relative participation of the former poles to the input–output frequency response of the plant/filter cascade connection.

In this paper we present a method to design cascade compensators when there is frequency overlap between the poles we wish to control and any other plant poles. Frequency overlap takes place, for example, when there are several modes with similar natural frequencies but only a few of them require damping augmentation. In such a case it is not possible to use frequency selective filters to direct the control energy to the relevant modes only.

The focus is on static cascade compensators with no dynamic elements. The goal is to design cascade compensators that increase the relative contribution of a specified subset of plant poles to the input–output frequency response of the cascade connection of the plant with the compensator. The paper shows how such compensators may be obtained from the generalized singular value decomposition (GSVD) of two matrices constructed with frequency response data. The GSVD is used to separate the plant poles in space rather than in frequency. Thus, static cascade compensators that achieve the stated goal exist only if there are multiple control inputs, multiple measured outputs, or both.

The design of static precompensators is described in Sec. II. This section contains a detailed example that illustrates the method and its role in controller design. Section III treats the problem of postcompensator design. Conclusions are given in Sec. IV. The GSVD and its main properties are in the Appendix.

Mario Rotea received his electronic engineering degree from the Universidad Nacional de Rosario (UNR), Argentina, in 1983 and his M.S. in electrical engineering and Ph.D. in control science and dynamical systems from the University of Minnesota in 1988 and 1990, respectively. From 1983 to 1984, Dr. Rotea was an Assistant Engineer at the Military Ammunition Factory Fray Luis Beltran (Argentina) working in software development for programmable logic controllers. From 1984 to 1986, he was a Research Associate at the Institute of Technological Development for the Chemical Industry (Argentina) working primarily in the robust process control. In 1990, Dr. Rotea joined Purdue University, West Lafayette, where he is an Associate Professor of Dynamics and Control. In 1997–1998, Dr. Rotea was a Senior Research Engineer with United Technologies Research Center, Connecticut, working on the development of practical solutions for identification and control of turbomachinery, manufacturing processes and machine tools, and helicopter systems. In 1993, Dr. Rotea was awarded a National Science Foundation Young Investigator Award. Dr. Rotea has authored over 70 publications in control and optimization theory and applications.

Received 24 February 1999; revision received 9 September 1999; accepted for publication 9 September 1999.

II. Design of Static Precompensators

We construct cascade compensators that increase the contribution to the compensated frequency response of a specified subset of plant poles relative to the contribution of any other plant poles. (Note that the compensated frequency response is that of the series connection of the cascade compensator and the plant.) This is accomplished using the GSVD of a matrix pair formed with frequency response data; see Result 1 in the Appendix for the definition of the GSVD. A detailed example demonstrating the results in a typical control problem is given also. The focus of this section is on precompensators; postcompensators are treated separately in the next section.

Let $P(s)$ denote the plant transfer matrix. Assume that $P(s)$ is a $p \times m$ rational matrix function. Take a partial fraction expansion of $P(s)$ of the form

$$P(s) = P_1(s) + P_2(s) \tag{1}$$

Assume that $P_2(s)$ contains only the plant poles whose contribution to the compensated frequency response we want to increase (e.g., the poles whose dynamics we wish to control), and that $P_1(s)$ contains all of the other plant poles.

Our goal is to find a constant matrix W such that the frequency response $P_2(J\omega)W$ dominates the frequency response $P_1(J\omega)W$. Ideally, we want W such that the approximation

$$P(J\omega)W = [P_1(J\omega) + P_2(J\omega)]W \approx P_2(J\omega)W \tag{2}$$

holds for all ω in a frequency interval containing the poles of $P_2(s)$.

To give precise meaning to the approximation in Eq. (2), we need to measure how big is $P_1(J\omega)W$ relative to $P_2(J\omega)W$ on a given set of frequencies. This can be accomplished by evaluating the largest generalized singular value (GSV) of a suitable constructed matrix pair. The largest GSV of a matrix pair is denoted by $\sigma_{\max}(\cdot,\cdot)$ and it is defined in the Appendix, Eq. (A4).

Let $\omega_1, \ldots, \omega_N$ denote given positive frequencies. Define the $2Np \times m$ real matrices

$$M_1 \stackrel{\text{def}}{=} \begin{bmatrix} \Re\{P_1(J\omega_1)\} \\ \vdots \\ \Re\{P_1(J\omega_N)\} \\ \Im\{P_1(J\omega_1)\} \\ \vdots \\ \Im\{P_1(J\omega_N)\} \end{bmatrix}, \qquad M_2 \stackrel{\text{def}}{=} \begin{bmatrix} \Re\{P_2(J\omega_1)\} \\ \vdots \\ \Re\{P_2(J\omega_N)\} \\ \Im\{P_2(J\omega_1)\} \\ \vdots \\ \Im\{P_2(J\omega_N)\} \end{bmatrix} \tag{3}$$

where $P_1(s)$ and $P_2(s)$ are from Eq. (1). Given $W \in \mathbb{R}^{m\times q}$, we propose to measure the average size of $P_1(J\omega)W$ relative to the average size of $P_2(J\omega)W$, on the set $\{\omega_1, \ldots, \omega_N\}$, with $\sigma_{\max}(M_1W, M_2W)$.

Justification for $\sigma_{\max}(M_1W, M_2W)$ as an appropriate measure of relative size comes from Result 3 (see Appendix) and simple algebraic manipulations. Indeed, from the definitions in Eq. (3) and the identities

$$M_i' M_i = \sum_{k=1}^{k=N} \Re\left\{P_i^*(J\omega_k)P_i(J\omega_k)\right\} = \frac{1}{2}\sum_{k=-N}^{k=+N} P_i^*(J\omega_k)P_i(J\omega_k)$$

where we define $\omega_{-k} \stackrel{\text{def}}{=} -\omega_k$, it follows from Result 3, item 1, that $\sigma_{\max}(M_1W, M_2W)$ is precisely the smallest number γ that satisfies the matrix inequality

$$\sum_{k=-N}^{k=+N} W'P_1^*(J\omega_k)P_1(J\omega_k)W \le \gamma^2 \sum_{k=-N}^{k=+N} W'P_2^*(J\omega)P_2(J\omega)W \tag{4}$$

Hence, $\sigma_{\max}(M_1W, M_2W) < 1$ implies that, on the average, $P_2(J\omega)W$ dominates $P_1(J\omega)W$ on the set $\{\omega_1, \ldots, \omega_N\}$. In fact, if $\sigma_{\max}(M_1W, M_2W) \ll 1$, the approximation in Eq. (2) will hold.

From a control standpoint, a matrix W such that the approximation in Eq. (2) holds will help direct the available control energy to the poles of $P_2(s)$ (the plant poles whose dynamics we wish to control). Such a matrix W will also help reduce the energy spillover to the poles of $P_1(s)$. Intuitively, Eq. (2) implies that the poles of $P_1(s)$ are not controllable from the inputs of $P(s)W$ or not observable from its outputs. Hence, any feedback loop around $P(s)W$ will affect mainly the poles of $P_2(s)$.

A. Computation of an Optimal Precompensator

From the preceding discussion it follows that the optimization problem

$$\inf_{W\in\mathbb{R}^{m\times q}} \sigma_{\max}(M_1W, M_2W) \tag{5}$$

$$\text{subject to rank } W = q$$

gives a sensible method for computing W. Even if the optimal largest GSV is greater than one, a solution to Eq. (5) will minimize the average size of $P_1(J\omega)W$, relative to the average size of $P_2(J\omega)W$, on the specified set of frequencies. A solution to the optimization problem in Eq. (5) is given in the Appendix, Result 3, item 2. The following design procedure summarizes the computations required to obtain this solution.

Pick a set of frequencies $\{\omega_1, \ldots, \omega_N\}$, that covers the magnitudes of the poles of $P_2(s)$ and the nearby poles of $P_1(s)$ and do the following.

1) Compute the matrices M_1 and M_2 defined in Eq. (3).

2) Compute the GSVD of the pair (M_1, M_2) and the matrix X defined in Result 1 (see Appendix). Let x_j denote the jth column of the matrix X.

3) Pick an integer $q \le m$; a solution to the optimization problem in Eq. (5) is given by any full-rank matrix $W_q \in \mathbb{R}^{m\times q}$ whose columns span the q-dimensional subspace

$$\text{im}\,[x_1 \quad x_2 \quad \cdots \quad x_q]$$

and the optimal GSV is $\sigma_{\max}(M_1W_q, M_2W_q) = \sigma_q(M_1, M_2)$.

In applications, the matrices M_1 and M_2 will typically have many more rows than columns. In such cases, the condition rank $[M_1' \;\; M_2'] = m$ is generic. Per Result 1 (see Appendix), the GSVD of the pair (M_1, M_2) will generically exist.

There is no need to calculate W_q when $q = m$. Per step 3 $W_m = I$ is optimal when $q = m$. The case $q = m$ makes sense only if $\sigma_{\max}(M_1, M_2) \ll 1$; see the next paragraph.

The success of the design procedure depends on the GSVs of the pair (M_1, M_2) and the value selected for q. We have no control over the GSVs; we can only play with q to achieve the best result. The selection of q is problem dependent; some general guidelines are as follows. The integer q is the number of degrees of freedom available to control the poles of $P_2(s)W_q$; thus, q large is desirable. Compute all of the GSVs of (M_1, M_2) and take q to be the largest integer such that $\sigma_q(M_1, M_2) < 1$. Alternatively, take q to be the largest integer such that $\sigma_q(M_1, M_2) \ll \sigma_{q+1}(M_1, M_2)$. Even if $\sigma_q(M_1, M_2) > 1$, this latter selection rule will produce a precompensator W_q that greatly reduces the average size of $P_1(J\omega)W_q$, relative to the average size of $P_2(J\omega)W_q$, over the frequencies of interest.

When there is a single control input ($m = 1$), the only choice is $q = 1$. As already explained, $q = m = 1$ implies that the scalar precompensator $W_1 = 1$ is optimal. Therefore, in the single input case, a trivial precompensator always results. Only if there are multiple plant outputs ($p > 1$), a postcompensator (see Sec. III) may perform better.

B. Example

Consider a plant with two control inputs and one sensor output given by

$$P(s) = \sum_{h=1}^{3} \frac{N_h(s)}{(s+\sigma_h)^2 + \mu_h^2} \tag{6}$$

where the parameters are defined in Table 1.

Table 1 Parameters of the transfer matrix $P(s)$ in Eq. (6)

h	σ_h	μ_h	N_h (input 1)	N_h (input 2)
1	0.0765	0.8466	$0.0823\,(s + 0.4927)$	$0.0393\,(s + 3.2394)$
2	0.0100	1.0000	$0.1349\,(0.7669 - s)$	$0.0952\,(s - 0.2967)$
3	0.1150	1.1443	$0.0832\,(3.9109 - s)$	$0.0310\,(s + 20.8436)$

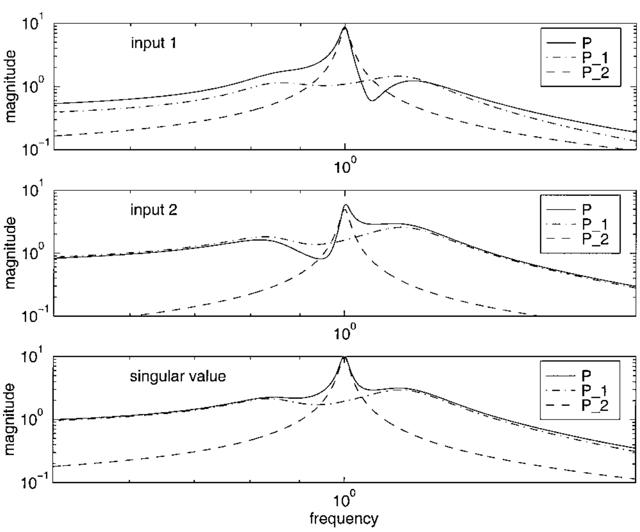

Fig. 1 Frequency responses of $P(s)$ and its two partial fractions $P_1(s)$ and $P_2(s)$.

The problem is to find a feedback controller $K(s)$ that increases the minimal damping ratio of the system, which is about 1% in the second mode ($h = 2$). We solve this problem using the following procedure: 1) Compute a suitable precompensator W_q. 2) Design a controller for the shaped plant $P(s)W_q$.

The damping ratios of the first and third modes are eight times (at least) the damping ratio of the second mode (see Table 1). Our goal is to design a feedback controller that augments the damping of the second mode ($h = 2$) and spends little or no energy on the other modes.

Decompose the plant $P(s)$ as shown in Eq. (1). The transfer matrix $P_1(s)$ contains the terms corresponding to $h = 1$ and $h = 3$ in Eq. (6). The transfer matrix $P_2(s)$ contains the term corresponding to $h = 2$, which is the mode we wish to control. Figure 1 shows the magnitude of the frequency responses $P(J\omega)$, $P_1(J\omega)$, and $P_2(J\omega)$ for ω in the range 0.5–2. All three modes show up in $P(J\omega)$ (solid line) regardless of the combination of inputs used to actuate the system (input 1, input 2, or both inputs).

Using the design procedure from the preceding subsection, we may determine a precompensator that increases the relative contribution of the second mode to the frequency response. The plant natural frequencies range from μ_1 to μ_3 (see Table 1). To compute the pair (M_1, M_2) defined in Eq. 3, we take 50 equally spaced points $\omega_1, \ldots, \omega_{50}$ in the range $0.9\mu_1$–$1.10\mu_3$. The matrices M_1 and M_2 have 100 rows and 2 columns; thus, there are two GSVs given by

$$\sigma_{\min}(M_1, M_2) = 0.1083, \qquad \sigma_{\max}(M_1, M_2) = 5.9037$$

Because $\sigma_{\max}(M_1, M_2) > 1$, we conclude that $P_2(J\omega)$ does not dominate $P_1(J\omega)$. This conclusion is consistent with the frequency responses shown in Fig. 1 that demonstrate that all modes show up in the frequency response $P(J\omega)$.

Notice, however, that $\sigma_{\min}(M_1, M_2) \ll 1$. Hence, we conclude that there is a rank-one matrix $W_1 \in \mathbb{R}^{2\times 1}$ such that $P_2(J\omega)W_1$ strongly dominates $P_1(J\omega)W_1$. To compute W_1, we need the matrix X in the GSVD of the pair (M_1, M_2). This matrix is given by

$$X = \begin{bmatrix} -0.0507 & 0.0305 \\ 0.0301 & 0.0539 \end{bmatrix}$$

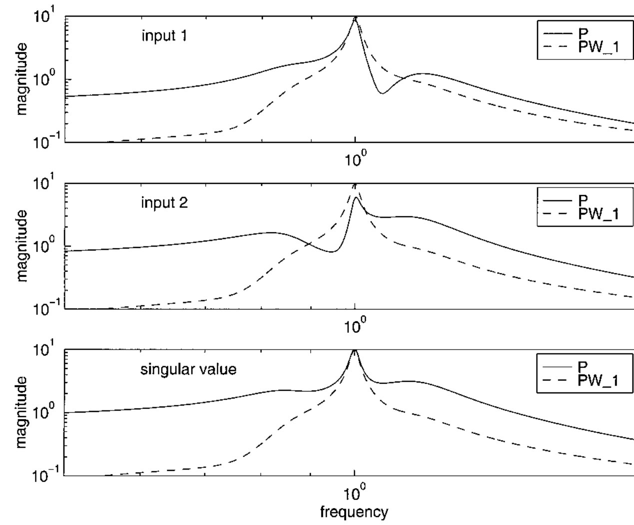

Fig. 2 Frequency responses of $P(s)$ and the precompensated plant $P(s)W_1$.

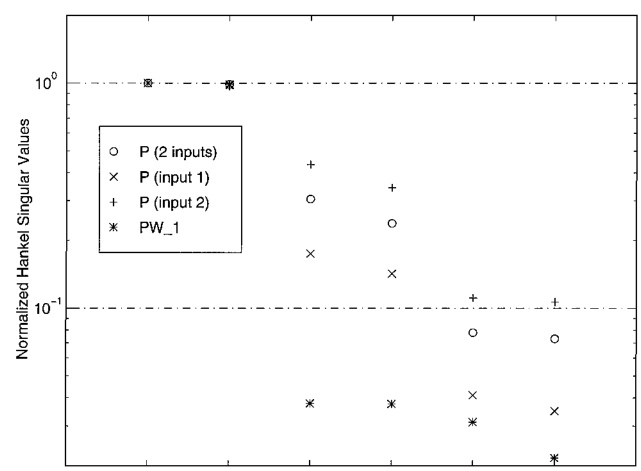

Fig. 3 Hankel singular values of $P(s)$ and the precompensated plant $P(s)W_1$.

Normalizing the first column of X gives the optimal rank-one precompensator

$$W_1 = \begin{bmatrix} -0.8601 \\ 0.5102 \end{bmatrix} \tag{7}$$

Figure 2 shows the magnitude of the frequency responses $P(J\omega)$ (solid line) and $P(J\omega)W_1$ (dashed line). The resonance at the frequency of the second mode (unit frequency) is much more noticeable in the shaped plant $P(s)W_1$ than in the original plant $P(s)$. The ability of the precompensator W_1 to emphasize the second mode, over the other two modes, can also be seen in Fig. 3. Figure 3 shows the Hankel singular values of $P(s)$, with both inputs acting simultaneously and also with one active input at a time, as well as the Hankel singular values of the shaped plant $P(s)W_1$. In all four cases, the singular values are normalized so that the largest singular value is exactly one. Notice the reduction in the last four Hankel singular values achieved by W_1. This reduction is so significant that $P(s)W_1$ has only two controllable/observable poles. It can be verified, for example, by computing a second-order balanced truncation, that these poles correspond to the poles of $P_2(s)$ (the poles we wish to control).

Next, we design a feedback compensator that increases the minimal (open-loop) damping ratio using the shaped plant $P(s)W_1$. This step can be performed using any method. Here we use the loop shaping design procedure of McFarlane and Glover[1] because it is simple to use and effective for solving this class of problems.

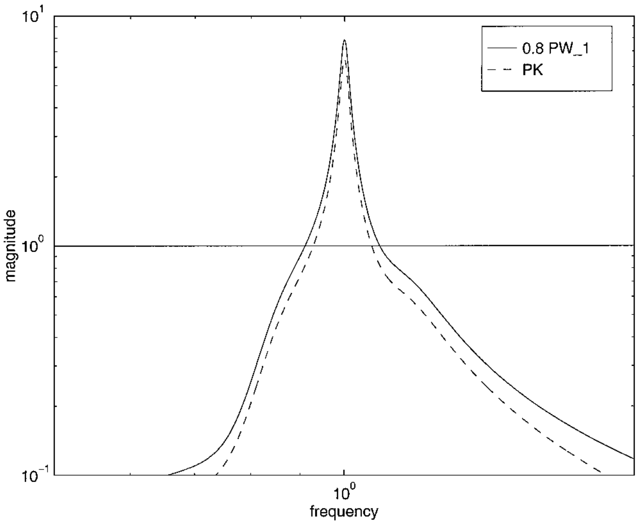

Fig. 4 Frequency responses of the target loop transfer function $P(s)W_1\gamma$ and the final loop transfer function at plant output $P(s)K(s)$.

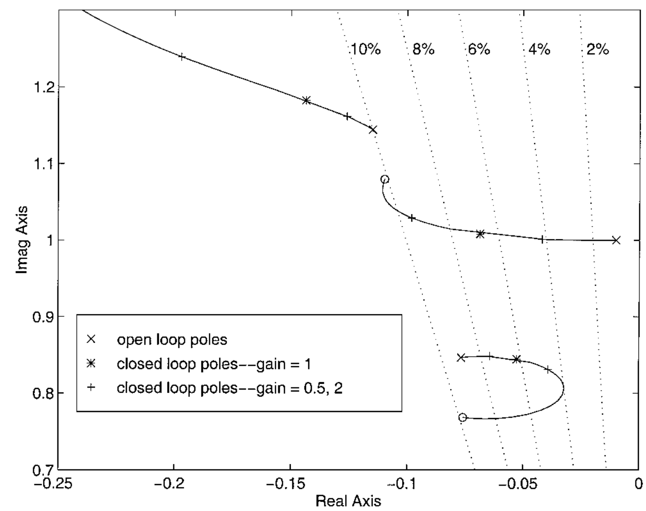

Fig. 5 Variation of closed-loop poles with the gain parameter.

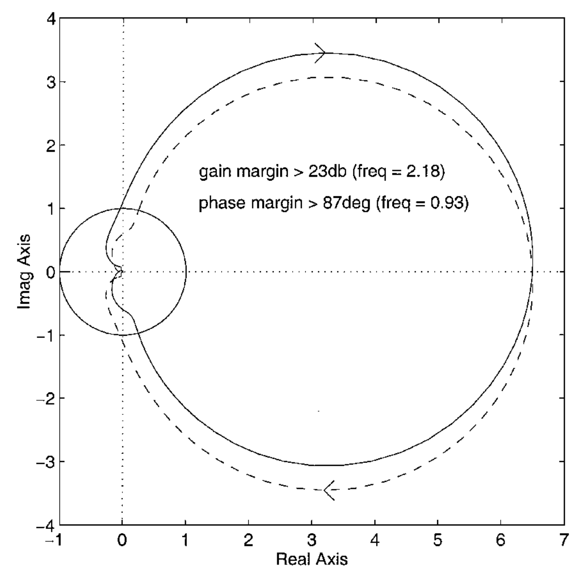

Fig. 6 Nyquist plot of $-P(s)K(s)$.

In this example, the following approach suffices. First, select a gain factor γ so that the magnitude of $P(j\omega)W_1\gamma$ is larger than one in a neighborhood of $\omega = 1$ (the frequency of mode we wish to control). Then, compute a dynamic feedback compensator $C(s)$ by solving a specific robust stabilization problem[1] for the shaped plant $P(s)W_1\gamma$ or for a reduced-order design model of $P(s)W_1\gamma$. The final controller for the plant $P(s)$ is given by $K(s) = W_1\gamma C(s)$.

Typically, the minimal closed-loop damping ratio increases with the gain factor γ. However, as γ increases more control authority is required. The solid line in Fig. 4 shows the magnitude of the loop gain $P(j\omega)W_1\gamma$ with $\gamma = 0.8$. The target loop gain is higher than one around $\omega = 1$. Thus, $\gamma = 0.8$ is deemed adequate; the design proceeds with this value for γ.

The Hankel singular values shown in Fig. 3 suggest that only the second mode ($h = 2$) needs to be retained in the transfer function of the shaped plant $P(s)W_1\gamma$. Performing model reduction of the normalized coprime factors of $P(s)W_1\gamma$ gives the second-order design model

$$G_r(s) = 0.1446\frac{s - 0.5885}{(s + 0.0098)^2 + 1.0042^2}$$

[Note that truncation of the balanced realization of $P(s)W_1\gamma$ is not advisable because the gap in the Hankel singular values of $P(s)W_1\gamma$ is independent of the desired performance level γ.] Notice that the poles of $G_r(s)$ are (practically) the poles we want to control; see the row corresponding to $h = 2$ in Table 1.

Solving the robust stabilization problem proposed by McFarlane and Glover,[1] with $G_r(s)$ as the nominal design model, gives the following optimal solution:

$$C(s) = 0.9632\frac{s - 3.3868}{s + 4.0136}$$

To get a more practical solution, we augment $C(s)$ with a low-pass filter (cutoff frequency equal to 2) and with a high-pass filter (cutoff frequency equal to 0.5) in order to increase rolloffs outside the frequency range 0.5–2. With $C(s)$ modified in this way, the formula $K(s) = W_1\gamma C(s)$ gives the final controller

$$K(s) = \begin{bmatrix} -1.6568 \\ 0.9828 \end{bmatrix} \frac{s(s - 3.4)}{(s + 0.5)(s + 2)(s + 4)} \tag{8}$$

The performance and robustness of the feedback system with the controller $K(s)$ in Eq. (8) is analyzed next.

The root-locus of the characteristic equation $1 - \text{gain}\ P(s)K(s) = 0$ is shown in Fig. 5. Only the variation of the poles of $P(s)$ is shown. [Note that with gain $\lesssim 2$, the branches emanating from the poles of $K(s)$ have damping greater than 98% and are of no importance to the problem.] The designed controller (gain = 1) achieves more than 6% damping in all modes. This number represents more than six times the minimal open-loop damping ratio. The damping in the second mode is increased from 1% to about 7%. As expected, the designed controller affects mostly the damping of the second mode. If the loop gain is reduced/increased by a factor of two, the minimal closed-loop damping ratio exceeds 4%, which shows that performance is robust to nontrivial gain variations.

The dashed line in Fig. 4 shows the magnitude of $P(j\omega)K(j\omega)$; this response is very close to the target response (solid line). The Nyquist plot of $P(j\omega)K(j\omega)$ is shown in Fig. 6. Excellent margins are achieved. Figure 7 shows the complementary sensitivities at plant output $(1 - PK)^{-1}PK$ and plant input $(I - KP)^{-1}KP$. Notice from the solid line that, at plant output, the effect of sensor noise is attenuated for all frequencies.

Mu-tests[2] were performed to determine the robustness of stability to simultaneous perturbations in the plant input and output channels and in the plant natural frequencies and damping ratios. The uncertain plant used for the mu-tests is

$$(1 + \Delta_o)P_{\text{per}}(s)\ \text{diag}(1 + \Delta_{i1}, 1 + \Delta_{i2}) \tag{9}$$

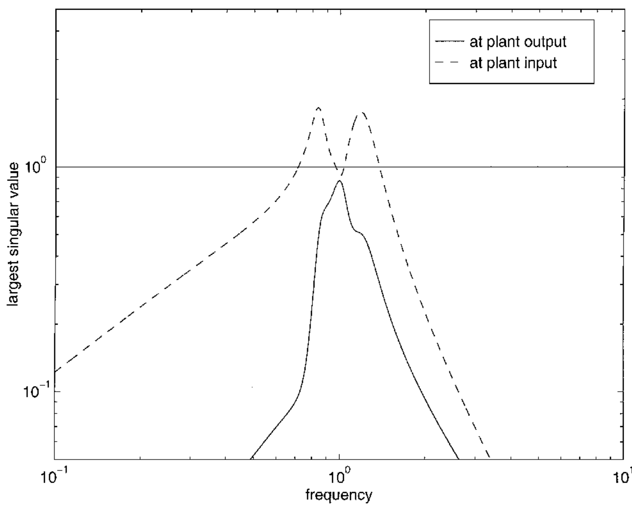

Fig. 7 Complementary sensitivities with $K(s)$.

The parameters Δ_o, Δ_{i1}, and Δ_{i2}, are complex scalars, and they represent multiplicative uncertainty in the input and output channels. The transfer matrix $P_{\mathrm{per}}(s)$ is obtained from Eq. (6) by perturbing the nominal (undamped) natural frequencies and damping ratios with multiplicative real uncertain parameters. The final controller in Eq. (8) was connected to Eq. (9) to determine the robustness of stability to these simultaneous perturbations.

In the first mu-test, Δ_o, Δ_{i1}, and Δ_{i2}, are the only sources of uncertainty. Closed-loop stability is maintained provided $|\Delta_o| \leq 36.6\%$, $|\Delta_{i1}| \leq 36.6\%$, and $|\Delta_{i2}| \leq 36.6\%$. If less variation is allowed in Δ_{i1} and Δ_{i2}, stability is guaranteed for simultaneous perturbations satisfying $|\Delta_o| \leq 58.6\%$, $|\Delta_{i1}| \leq 28.4\%$, and $|\Delta_{i2}| \leq 28.4\%$. In the second mu-test, the three natural frequencies and damping ratios are also allowed to vary. Closed-loop stability is maintained provided $|\Delta_o| \leq 37.8\%$, $|\Delta_{i1}| \leq 18.4\%$, $|\Delta_{i2}| \leq 18.4\%$, and the three natural frequencies, and damping ratios change by no more than 13.2 and 24.4%, respectively. If the size of Δ_o, Δ_{i1}, and Δ_{i2} does not exceed 14.2%, then uncertainty in damping ratios can be increased to 42.6% without losing stability.

III. Design of Static Postcompensators

The objective of this section is to describe how to compute a suitable postcompensator. This means finding W such that $WP_2(J\omega)$ dominates $WP_1(J\omega)$, where $P_1(s)$ and $P_2(s)$ are from Eq. (1).

Let $\omega_1, \ldots, \omega_N$ denote given positive frequencies. Define the real matrices

$$N_1 \stackrel{\mathrm{def}}{=} [\Re\{P_1(J\omega_1)\} \quad \cdots \quad \Re\{P_1(J\omega_N)\} \quad \Im\{P_1(J\omega_1)\} \quad \cdots \quad \Im\{P_1(J\omega_N)\}] \tag{10}$$

$$N_2 \stackrel{\mathrm{def}}{=} [\Re\{P_2(J\omega_1)\} \quad \cdots \quad \Re\{P_2(J\omega_N)\} \quad \Im\{P_2(J\omega_1)\} \quad \cdots \quad \Im\{P_2(J\omega_N)\}] \tag{11}$$

Let $W \in \mathbb{R}^{r\times p}$ be given, where p is the number of rows of $P(s)$. Then, from Result 3, item 1 (see Appendix), we may conclude that $\sigma_{\max}(N_1'W', N_2'W')$ is precisely the smallest number γ that satisfies the matrix inequality

$$\sum_{k=-N}^{k=+N} WP_1(J\omega_k)P_1^*(J\omega_k)W' \leq \gamma^2 \sum_{k=-N}^{k=+N} WP_2(J\omega)P_2^*(J\omega)W' \tag{12}$$

where $\omega_{-k} \stackrel{\mathrm{def}}{=} -\omega_k$. Hence, $\sigma_{\max}(N_1'W', N_2'W')$ measures the average size of $WP_1(J\omega)$, relative to the average size of $WP_2(J\omega)$, on the set $\{\omega_1, \ldots, \omega_N\}$.

From this discussion, it follows that the optimization problem

$$\inf_{W\in\mathbb{R}^{r\times p}} \sigma_{\max}(N_1'W', N_2'W') \quad \text{subject to rank } W = r \tag{13}$$

gives a sensible method for computing an optimal postcompensator W. Moreover, our design procedure delivers an optimal solution to Eq. (13) provided M_1, M_2, q, and W are replaced by N_1', N_2', r, and W', respectively. With the obvious modifications, the remarks on the use of the design procedure for the computation of precompensators apply to the case of postcompensators as well.

IV. Conclusions

A method for the computation of static cascade compensators that increase the contribution to the input–output frequency response of a specified subset of plant poles relative to the contribution of the remaining plant poles has been described. The computations required to implement the results are a partial fraction expansion of the plant transfer matrix and the GSVD of a matrix pair constructed with frequency response data. These two calculations are rather simple.

The results are useful when one needs to design a controller that modifies the dynamics associated with a particular subset of poles, and at the same time, the controller has little effect on the remaining poles. Indeed, once a suitable cascade compensator has been determined, robust controller design methods can be used to complete the design in a simple and effective manner.

Both precompensators and postcompensators have been considered. The final selection between these two alternatives is problem dependent. Fortunately, the required calculations are so simple that both alternatives could be explored with very little design effort.

Appendix: GSVD

The following result is a special case of Theorem 8.7.4, Ref. 3, p. 466, by Golub and Van Loan. See also the work of de Moor.[4]

Result 1 [GSVD of the pair (M_1, M_2)]: Let $M_1 \in \mathbb{R}^{n_1\times m}$ and $M_2 \in \mathbb{R}^{n_2\times m}$ denote two real matrices with the same number of columns. Assume that $\min(n_1, n_2) \geq m$ and rank $[M_1' \ M_2'] = m$. There exist $U_1 \in \mathbb{R}^{n_1\times n_1}$, $U_2 \in \mathbb{R}^{n_2\times n_2}$, $C \in \mathbb{R}^{n_1\times m}$, $S \in \mathbb{R}^{n_2\times m}$, and an invertible $X \in \mathbb{R}^{m\times m}$ satisfying the following equations:

$$M_1 = U_1CX^{-1}, \qquad M_2 = U_2SX^{-1}, \qquad I = U_1U_1'$$

$$I = U_2U_2', \qquad I = C'C + S'S$$

$$C = \mathrm{diag}(c_1, \ldots, c_m), \qquad 0 \leq c_1 \leq c_2 \leq \cdots \leq c_m$$

$$S = \mathrm{diag}(s_1, \ldots, s_m), \qquad 0 \leq s_m \leq s_{m-1} \leq \cdots \leq s_1$$

The set $\{c_1/s_1, \ldots, c_m/s_m\}$ depends only on the matrices M_1 and M_2. The elements of this set are the GSVs of the pair (M_1, M_2), which we will denote by

$$\sigma_j(M_1, M_2) \stackrel{\mathrm{def}}{=} (c_j/s_j) \tag{A1}$$

If $M_2 = I$, $\{\sigma_1(M_1, M_2), \ldots, \sigma_m(M_1, M_2)\}$ is the set of singular values of M_1, for $X = U_2S$, and S is diagonal and invertible. Thus, Result 1 is a generalization of the SVD. The GSVD may be computed with the programs gsvd.m and csd.m in S. J. Leon's User-Contributed M-Files: gsvd.m and csd.m (URL: http://www.mathworks.com/linalgv4.html), which work if the assumptions in Result 1 hold.

From Result 1, it follows that the GSVs satisfy the inequality

$$0 < \sigma_1(M_1, M_2) \leq \sigma_2(M_1, M_2) \leq \cdots \leq \sigma_{m-1}(M_1, M_2) \leq \sigma_m(M_1, M_2) \tag{A2}$$

which justifies the definitions

$$\sigma_{\min}(M_1, M_2) \stackrel{\mathrm{def}}{=} \sigma_1(M_1, M_2) \tag{A3}$$

$$\sigma_{\max}(M_1, M_2) \stackrel{\text{def}}{=} \sigma_m(M_1, M_2) \tag{A4}$$

In this paper the GSVs are always arranged in increasing order [Eq. (A2)]. Rank $M_1 < m$ implies $\sigma_{\min}(M_1, M_2) = 0$, and rank $M_2 < m$ implies $\sigma_{\max}(M_1, M_2) = \infty$. Therefore, in the pair (M_1, M_2), we can think of M_1 as the numerator matrix and M_2 as the denominator matrix.

Result 2 (properties of the GSVD): Given any $j = 1, \ldots, m$, under the assumptions of Result 1, we have the following.

1) Here $\sigma_j(M_2, M_1) = \sigma^{-1}_{m-j+1}(M_1, M_2)$.

2) For any orthogonal matrices $Q_1 \in \mathbb{R}^{n_1 \times n_1}$ and $Q_2 \in \mathbb{R}^{n_2 \times n_2}$, and any nonsingular matrix $T \in \mathbb{R}^{m \times m}$, $\sigma_j(Q_1 M_1 T, Q_2 M_2 T) = \sigma_j(M_1, M_2)$.

3) $M_1' M_1 x_j = M_2' M_2 x_j \sigma_j^2(M_1, M_2)$, where x_j denotes the jth column of the matrix X in Result 1.

A proof of this result is obtained by direct verification using the equations in Result 1. Property 3 shows that GSVs are the square roots of the generalized eigenvalues of the symmetric matrix pair $(M_1' M_1, M_2' M_2)$. Hence, the GSVD will play a key role in solving certain optimization problems involving the quadratic forms $\|M_1 x\|^2$ and $\|M_2 x\|^2$. The following result summarizes the solution of the optimization problems of importance to this paper.

Result 3 (optimization of the ratio of quadratic forms): Consider the GSVD of the pair (M_1, M_2) as defined in Result 1, and partition the matrix X according to

$$X = [x_1 \quad x_2 \quad \cdots \quad x_{m-1} \quad x_m]$$

where $x_j \in \mathbb{R}^m$. The following statements hold.

1) The vector $\boldsymbol{x}_m$ and the largest GSV satisfy

$$\sup_{x \in \mathbb{R}^m} \frac{\|M_1 x\|}{\|M_2 x\|} = \frac{\|M_1 x_m\|}{\|M_2 x_m\|} = \sigma_{\max}(M_1, M_2)$$

2) Any full-rank matrix $W_q \in \mathbb{R}^{m \times q}$, whose columns span the q-dimensional subspace

$$\text{im}[x_1 \quad x_2 \quad \cdots \quad x_q]$$

and the qth GSV, satisfy

$$\inf_{W \in \mathbb{R}^{m \times q}} \sigma_{\max}(M_1 W, M_2 W) = \sigma_{\max}(M_1 W_q, M_2 W_q) = \sigma_q(M_1, M_2)$$

where the infimization is over all matrices W with rank q. Note that, under the assumptions of Result 1, rank $W'[M_1' \quad M_2'] = q$ for all full column rank matrices $W \in \mathbb{R}^{m \times q}$; thus, the GSVs of $(M_1 W, M_2 W)$ are well defined.

A proof of Result 3 is obtained from Result 2, Properties 2 and 3, and the well-known variational characterizations of eigenvalues as presented, for example, in Ref. 5.

Acknowledgment

This work was supported in part by the National Science Foundation under Young Investigator Award ECS-9358288, and in part by the United Technologies Corporation. The author is grateful to Fernando J. D'Amato for the mu-tests in the example section.

References

[1] McFarlane, D. C., and Glover, K., *Robust Controller Design Using Normalized Coprime Factorizations*, Lecture Notes in Control and Information Sciences, Vol. 138, Springer–Verlag, New York, 1990, pp. 98–131.

[2] Balas, G., Doyle, J. C., Glover, K., Packard, A., and Smith, R., "μ-Analysis and Synthesis Toolbox," MathWorks, Natick, MA, 1995.

[3] Golub, G. H., and Van Loan, C. F., *Matrix Computations*, 3rd edition, John Hopkins Univ. Press, Baltimore, MD, 1996, pp. 465–467.

[4] de Moor, B., "Generalizations of the Singular Value and QR Decompositions," *Signal Processing*, Vol. 25, No. 2, 1991, pp. 135–146.

[5] Horn, R. A., and Johnson, C. R., *Matrix Analysis*, Cambridge Univ. Press, New York, 1991, pp. 176–180.